

Theorem [3] Prove that  $(ab)^{-1} = b^{-1}a^{-1}$  where  $a, b \in G$ .

OR, prove that the inverse of the product of two elements of a group is the product of the inverses taken in reverse order.

Proof: Suppose  $a, b \in G$  and let their inverses be  $a^{-1}$  and  $b^{-1}$  respectively.

$$\begin{aligned} \text{Now } (b^{-1}a^{-1})(ab) &= b^{-1}\{a^{-1}(ab)\} \\ &= b^{-1}\{(a^{-1}a)b\} \\ &= b^{-1}(eb) = b^{-1}b \end{aligned}$$

$$\begin{aligned} \text{Similarly, } (ab)(b^{-1}a^{-1}) &= a\{b(b^{-1}a^{-1})\} \\ &= a\{(bb^{-1})a^{-1}\} \\ &= a\{(ea^{-1})\} \\ &= aa^{-1} \\ &= e \end{aligned}$$

Hence  $b^{-1}a^{-1}$  is the inverse of  $ab$ .

$$\text{Hence } (ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G.$$

Hence proved.

Note: - The rule given in this theorem is known as the reversal Law.

Theorem [4] If  $a$  and  $b$  are elements of a group  $G$ , then prove that the equations

- (i)  $ax = b$  and (ii)  $ya = b$

have unique solutions in  $G$ .

Proof: [i] Let us consider the equation

$$ax = b \quad (1)$$

We are then to show that  $a^{-1}b$  is the solution of the given equation.

Since  $a^{-1}$  and  $b \in G$ ,

therefore  $a^{-1}b \in G$ .

If  $a^{-1}b$  is the solution of the equation, then  $x = a^{-1}b$  must satisfy the given eqn. (1).

Now putting  $x = a^{-1}b$  in (1), we get

$$\begin{aligned} \text{L.H.S.} &= a(a^{-1}b) = (aa^{-1})b \\ &= eb \\ &= b. \end{aligned}$$

Therefore the equation has a solution  $x = a^{-1}b$ .

Now have to show that  $x = a^{-1}b$  is the unique solution.

If not, suppose  $x = c$  is another solution in  $G$ .

Putting  $x = c$  in (1), we get  $ac = b$ .

Multiplying both sides by  $a^{-1}$  on the left, we get

$$a^{-1}(ac) = a^{-1}b$$

$$\Rightarrow (a^{-1}a)c = a^{-1}b$$

$$\Rightarrow ec = a^{-1}b$$

$\Rightarrow c = a^{-1}b$ , which shows that whatever solution we assume for the equation, it will come out to be  $a^{-1}b$ .

Thus, we have proved that the solution  $x = a^{-1}b$  is unique.

We can easily prove the theorem (ii) in similar process.

Hence the proof